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A Contribution to the Theory of the General Equation of the Sixth Degree.

BY F. N. COLE.

The subject of the present article was suggested to me by Prof. Klein, when I was a student in his Seminar at Leipzig, and I wish here to acknowledge my great indebtedness to him for valuable advice and suggestion, which have been of the greatest use to me. The fundamental idea of the entire treatment of the subject is due to him, as I have indicated below, and he might claim many of the particular methods involved as his own, if he should consider them worthy of such recognition.

The modern theory of the equation of the fifth degree may be said to date from the demonstration by Abel of the impossibility of the algebraic solution of the general equation of the fifth degree,* and the establishment by the same mathematician of the algebraic equations for the division of the elliptic functions,† equations of which the general equation of degree five is essentially a particular case. With this latter discovery of Abel, two important steps were made. The actual solution of the general equation of the fifth degree was obtained, and its theory identified with that of certain elliptic functions whose nature was fast becoming known through the work of Abel and Jacobi, and, what was more important, the *formal constitution* of the equation was ascertained, and thereby an important impetus given to a theory then just struggling into existence, but now become one of the most powerful of mathematical instruments—the Theory of Substitutions. Fifty years before Abel, Lagrange‡ had succeeded in unifying the theories of the equations of the first four degrees, which, before his time, had been a set of detached facts rather than a theory.

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* Crelle's Journal, Bd. I.

† Abel: Oeuvres completes. Edition of Helmboe, 1881.

‡ Mem. d. Berliner Akademie 1770-1. Oeuvres, T. III.

Encouraged by his success in this direction, he proposed a similar theory for the algebraic solution of the general equation of any degree. This theory, although it necessarily failed, contained the germs from which, later, the Theory of Substitutions developed. It was Evariste Galois* who understood how to reject what was erroneous in Lagrange's theory, and to adapt the remainder to the increased mathematical knowledge of his own day. He had succeeded in advancing the theory so far that he was able to make a direct application of it to this new discovery of Abel's. On this double foundation of the theory of the elliptic functions and the Theory of Substitutions, the theory of the equation of the fifth degree is based. Had Galois lived a few years longer, he would undoubtedly have constructed a tolerably complete theory of this and other equations, but, as it is, this theory has been one of slow growth. It may now be regarded as complete, and further investigation naturally takes a turn in the direction of the theory of the higher equations.

To Hermite† and Kronecker‡ we owe the greater part of our knowledge of the general equation of the fifth degree. The former started from the equation of the sixth degree, which Abel deduced for the division of the elliptic functions by 5, and obtained the general equation of the fifth degree as a resolvent of this. The latter operated with Jacobi's multiplier equation, which is also a resolvent of the equation of the fifth degree. For the further history of this subject, I refer to Prof. Klein's§ work on the Ikosahedron, which I shall frequently have occasion to cite, and which abounds in historical information, and is particularly rich in footnotes. To this work, too, I refer for the elegant geometrical developments which in late years Klein has given to this theory, connecting it with the modern geometrical methods, and giving it a completeness which was wanting in the purely analytical theory.

While the theory of the equation of the fifth degree is thus complete in all directions, that of the sixth degree is only just begun. Two methods of solution have been proposed, both analogous to the method for the fifth degree, but essentially different from each other in their points of view. Camille Jordan||

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* Evariste Galois: Oeuvres mathematiques. Edited by Liouville in T. XIV of Liouville's Journal.

† Hermite: Sur la résolution de l'équation du cinquième degré. C. R., t. 46 and 62.

‡ Kronecker: Monatsberichte der Berliner Akademie, 1861. Crelle, Bd. 59.

§ Klein: Vorlesungen über das Ikosaeder u. die Auflösung der Gleichungen vom fünften Grade. Teubner: Leipzig, 1884.

|| Traité des Substitutions, p. 380.

has shown that any algebraic equation of degree n may be solved by the aid of the formulae for the bisection of the periods of the corresponding hyperelliptic functions. This method would require the establishment of these formulae, and the reduction of the equation of the sixth degree to the form of a resolvent of them. This, so far as I know, has not yet been done. The second method was proposed by Klein,* and it is this method with which the present article has to deal. On account of the analogy of the method to that of the equations of lower degree, I prefix a short, and necessarily incomplete, account of the latter, referring to the original articles cited in the footnotes for complete information.

The central feature of the theories of the equations of degree one to five is the important part played by certain groups of linear transformations of a single variable. In fact, the theory of these groups is, formally, the theory of the corresponding equations, and appears, under one metamorphosis or another, in all the various phases in which their theory can be presented. The fourth degree is the first for which the group of linear transformation is of prominence. Here it is composed of the six linear transformations of the anharmonic ratio of the roots. The four roots have six anharmonic ratios, and, if one of them be denoted by λ, the others, as is well known, are:† $\frac{1}{\lambda}$, $1-\lambda$, $\frac{1}{1-\lambda}$, $\frac{\lambda}{1-\lambda}$, $\frac{\lambda-1}{\lambda}$, all of which proceed from any one by linear transformation. These six anharmonic ratios satisfy an equation of the sixth degree, which, of course, is not the general equation, but is characterized by the fact that every root is a rational linear function of every other root. This is the equation of the sixth degree which appears in the ordinary theory of the solution of the equation of the fourth degree; it reduces at once to a cubic which may then be solved, and from these solutions those of the biquadratic are at once obtained.

For the equation of the fifth degree, the matter is more complicated. Here the group of linear transformations contains sixty operations, which are represented as follows: If we denote by z a number congruent to $0, 1, 2, 3, 4$, or ∞ mod. 5, and from $z' \equiv \frac{\alpha z+\beta}{\gamma z+\delta}$ mod. 5, then, by proper choice of α, β, γ, δ, z' will be an integer, whichever quantity we take z congruent to. It is found that there are 60 sets of values, α, β, γ, δ, which satisfy these conditions.‡ If we select any one of these sets, and put z successively congruent to $0, 1, \ldots . 4, \infty$ mod. 5, we shall get for z' values congruent to $0 \ldots . \infty$, but in a different order from the original. Now, a rational function of the roots of the equation of the fifth degree can be found,* which, when these roots undergo any *even* permutation, is linearly transformed by the formula $\phi' = \frac{\alpha\varphi + \beta}{\gamma\varphi + \delta}$. This function ϕ then plays to a certain extent the same part in the theory of the equation of the fifth degree which the anharmonic ratio λ plays in that of the fourth degree. It satisfies an equation of degree 60, of which the coefficients involve no further irrationality than the square root of the discriminant of the given equation. There is, however, the important difference that the equation in λ is simpler than the general equation of the fourth degree, while that in ϕ is precisely the equivalent of the general equation of the fifth degree, and is neither more difficult nor simpler. In the language of the theory of substitutions, λ belongs to the exceptional self-conjugate† sub-group of the general group of 24 permutations of the 4 roots, the result being that the group of the λ equation is reduced to 6 permutations, while the ϕ equation has still the group of 60 subst. which the general equation of the fifth degree has after the adjunction of the square root of the discriminant.

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* Ikosaeder, p. 126, footnote. Math. Ann. XV.

† See, say, Clebsch: Geometrie, p. 39.

‡ Netto: Substitutionentheorie, p. 141.

If the z's above be taken as suffixes of six quantities x, every permutation of the z's will give a corresponding permutation of the x's. The resulting group of 60 substitutions could easily be identified with the monodromic‡ group of the modular equation of Abel and Hermite for the division of the elliptic functions by 5, but this would be beyond the scope of the present article. For the same reason I can only make passing mention here of the identification of the group of linear substitutions of the ϕ's with the group of rotations of the ikosahedron, and the connection with the line geometry of the surfaces of the second order.§

Of the greatest importance, on the other hand, for the present purpose, is the theory of the linear differential equations of the second and third orders given in the third chapter of Klein's work. We have seen that the quantities

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* Klein : Ikosaeder, p. 187, where λ and μ are such functions.

† I have ventured to translate Klein's "ausgezeichnete Untergruppe" by "self-conjugate sub-group," and use this nomenclature in preference to the "groupe permutable" of Jordan. For the theory of this group see in particular Klein's book, pp. 88-9.

‡ Camille Jordan : Traité, p. 277.

§ Cf. throughout Klein's book.

ϕ above satisfy an equation of degree 60 with coefficients which are rational after the adjunction of the square root of the discriminant. These coefficients may all be expressed as rational functions of a single parameter (Z of Klein's work). The roots of the equation are all obtainable from any one of these by linear transformation, $\phi' = \frac{\alpha\varphi + \beta}{\gamma\varphi + \delta}$, or $\zeta = \frac{\alpha\eta + \beta}{\gamma\eta + \delta}$, to follow Klein's notation. If this last equation be differentiated three times with respect to Z, we may eliminate α, β, γ and δ from the four resulting equations, and obtain the differential equation $\frac{\zeta'''}{\zeta'} - \frac{3}{2}\left(\frac{\zeta''}{\zeta'}\right)^2 = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$.* Either side of this equation is therefore an expression unchanged by any even permutation of the roots; and is therefore a rational function of Z. We have thus a differential equation which the quantities ϕ satisfy, and may study the solutions of this equation, from which those of the general equation of the fifth degree are rationally compounded.

The method proposed by Klein for the solution of the general equation of any degree is perfectly analogous to this. We have to seek a group of linear substitutions which shall be isomorphic† with the group of $n!$ permutations belonging to the equation. Functions of the roots must then be found which undergo these linear transformations when the roots are permuted; and finally, corresponding differential equations must be obtained and their solutions studied. In the present article, I have begun the determination of these differential equations for the case of the general equation of the sixth degree.

First of all, a group of linear transformations must be found which shall be isomorphic with the entire group of $6! = 720$ permutations of six things, or with the corresponding alternating group of 360 permutations. Here a short digression on the composition of the group of 720 permutations of six things, in which, as before, I merely indicate the sources of information, may not be out of place. First of all, this group contains the self-conjugate alternating sub-group of 360 permutations, for which the square root of the discriminant is the corresponding function. Next to this come, according to the law demonstrated by Bertrand and Serret,‡ the sub-groups of 120 permutations. By exception there are two of these, the ordinary intransitive group, affecting only five of the elements, and the extraordinary group containing only permutations of four, five and six elements.* As a result of the presence of this exceptional group, all equations of the sixth degree are connected in pairs, the roots of the two equations of each pair belonging respectively to the ordinary and the extraordinary groups of 120 permutations. There will be six of these groups of each kind. Half the permutations of each group will be odd and half even. The intransitive and imprimitive groups of lower orders are easily determined. The former are the system of sub-groups belonging to the general group of 5 elements. The latter are of orders which are divisors: (1) of 72, corresponding to the division of the six elements into two sets of three each; (2) of 48, for two sets, one of two, the other of four elements; and (3) also of 48, for three sets of two each. The primitive groups can contain no permutations which affect less than four elements.† They are to be determined according to the known methods.‡

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* Either side of this equation is the "Schwarzian Derivative" of Cayley. See Schwarz: Crelle, Bd. 75. Also Klein: Ikosaeder, p. 74, footnote.

† Jordan: Traité, p. 56.

‡ *Ib.* p. 67. Serret: Algèbre supérieur, Vol. II, Chap. 3.

If now we attempt to find a group of linear transformations which shall be isomorphic with the group of 720 or 360 permutations of six elements, we meet at the start with a certain difficulty. There is no such finite group of linear transformations of a single variable. The presence of such a linear group is a characteristic feature of the equations of the first five degrees, and seems to group these equations by themselves, as Klein has shown.§ Two ways offer themselves in which we can overcome the apparent difficulty arising from the absence of the linear group. On the one hand, we may look for infinite groups of linear transformations of a single variable; on the other, we may introduce additional variables and consider finite groups.|| The latter method is that with which we are to deal. We inquire, then, What is the smallest number of variables for which a group of linear substitutions isomorphic with the 720 permutations of six elements exists? There is no such group for one or for two variables. There is, however, such a group for three variables, or, if we write our linear transformation in homogeneous form, for four variables, of which the ratio of three to the fourth will then be transformed by a non-homogeneous linear transformation. This group of transformations is best known under the geometrical form in which its theory has been treated in connection with the

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* Serret: Alg. sup., Vol. II, Chap. 3. Netto: Substitutionentheorie, pp. 124-5.

† Netto, Chap. V.

‡ *Ib.* p. 125.

§ Klein, pp. 130-1.

|| *Ib.* Chap. V throughout.

remarkable surface of the fourth order and class known as Kummer's surface,[*] and with theory of the complexes of the second order of Plücker's Line Geometry.[†]

As is well known, the general equation of the second order in Plücker's line co-ordinates denotes a geometrical configuration composed of ∞^3 straight lines in space. These are so arranged that in every plane lie ∞^2 lines enveloping a conic section and through every point in space pass ∞^2 lines, forming a cone of the second order. One condition being necessary in order that a conic section should break up into two points, or that a cone of the second order should break up into two planes, there will be ∞^2 planes for which the conic is a pair of points and ∞^2 points for which the cone is a pair of planes. The surface enveloped by those ∞^2 planes and that traced out by the ∞^2 points are identical, and this surface, of which class and order are evidently equal, is Kummer's surface.[‡] Written in the ordinary line co-ordinates, the equation of the general complex of the second order is $\alpha_{12}p_{12}^2 + \alpha_{13}p_{13}^2 + \text{etc.} + \beta_{1234}p_{12}p_{34} + \text{etc.} = 0$, to which is conjoined the quadratic identity $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$. If we introduce the canonical co-ordinates of Klein into these equations, they become $\sum_{i=1}^{i=6} \varkappa_i x_i^2 = 0$ and $\Sigma x_i^2 = 0$, the cross-terms disappearing in both equations. The formulae connecting the two sets of equations may be taken as follows:

$$x_1 = p_{12} + p_{34},\ x_2 = -i(p_{12} - p_{34}),\ x_3 = p_{13} + p_{42},\ x_4 = -(p_{13} - p_{42}),$$
$$x_5 = p_{14} + p_{23},\ x_6 = -i(p_{14} - p_{23}).$$

If we denote the left-hand side of the equation of the complex by Ω, a linear tangential complex for the line y_i of the complex will be $\Sigma\left(\frac{d\Omega}{dyi}\right)x_i = 0$ or $\Sigma \varkappa_i x_i y_i = 0$. In order that the line y_i should be a singular line of the complex, we must have $\Sigma \varkappa_i^2 y_i^2 = 0$; *i. e.* the linear tangential complex must be a special one. The line y_i is then tangent to the Kummer surface, and the equation $\Sigma \varkappa_i x_i y_i = 0$ shows that every tangent to the surface at the point of contact of y_i will also cut $\varkappa_i y_i$. The bundle of tangents at this point is therefore represented by $\rho x_i = (\varkappa_i + \lambda) y_i$, where λ is an arbitrary variable parameter. These lines satisfy the equations $\Sigma x_i^2 = 0$, $\Sigma x_i y_i = 0$, $\Sigma y_i^2 = 0$; or, putting for y its value in terms of x, $\Sigma x_i^2 = 0$, $\Sigma \frac{x_i^2}{\varkappa_i + \lambda} = 0$, $\Sigma \frac{x_i^2}{(\varkappa_i + \lambda)^2} = 0$.

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[*] Kummer: Monatsberichte der Berliner Akademie, 1864.

[†] Plücker: Neue Geometrie des Raumes, etc. Edited by Clebsch and Klein.

[‡] For the geometry of Kummer's surface cf. throughout Rohn's article in Bd. XV and XVIII of the Math. Ann.

The first of these equations shows that x_i is a line; the second, that it belongs to a complex of the second order; the third, that it is a singular line of this complex. The Kummer surface belongs therefore to ∞' different complexes of the second order, corresponding to the ∞' different values of λ.

The six linear complexes $x_i = 0$ are called by Klein the fundamental complexes. A congruence of any two of them will consist of all lines which cut two fixed straight lines, the directrices of the congruence. If the six fundamental complexes be divided into three congruences, the six directrices will cut each other in pairs so as to form a tetrahedron. The 15 tetrahedra obtainable in this way are called the fundamental tetrahedra.

If, in the equation of the complex, we put for any x_i its negative, the equation will be unchanged. There are 32 possible arrangements of the signs, and these give rise to 32 transformations of space, which leave the Kummer surface unchanged. Sixteen of these are collineations, the other 16 being dualistic transformations. These convert in particular the system of 16 double points and 16 double tangent planes of the surface into the same system in a transposed order. Again, if we have used any system of co-ordinates based on any one of the 15 fundamental tetrahedra, these will be transformed if we take any other tetrahedron. Such a transformation amounts to a permutation of the fundamental complexes, *i. e.* of the x_i's. These permutations evidently leave Σx_i^2 unchanged. Every straight line therefore transforms into a straight line; that is, these transformations are either collineations or dualistic transformations. There are as many of them as there are permutations of the x_i's, *i. e.* 720. If we combine these with the 32 transformations above, we have a group of 32.720 linear transformations of space which is isomorphic with the group of the 720 permutations of six elements. This is the geometrical linear group to which I had reference above.*

We have next to determine a function of six arbitrary elements which is linearly transformed by this group of substitutions. To obtain such a function we observe that a permutation of the x's in this equation of our complex $\Sigma \kappa_i x_i^2 = 0$ amounts to a permutation of the κ's. Now, the point co-ordinates of a double point of the Kummer surface, referred to any one of the 16 fundamental tetrahedra, will be 16-valued functions of $\kappa_i \ldots . \kappa_6$. It appears, therefore, that the point co-ordinates of any double point of the Kummer surface referred to a fundamental tetrahedron are functions which undergo precisely the $\frac{32.720}{2}$ linear substitutions corresponding to the $\frac{32.720}{2}$ collineations of our group. These co-ordinates are the moduli whose theory was given by Borchardt in his article on the Arithmetic-Geometric Mean of Four Quantities.*

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*Cf. here Jordan: Traité, p. 317.

The moduli are 4 in number, corresponding to the 4 co-ordinates of the double point. They are given by the equations:

$$\xi^2 + \eta^2 + \zeta^2 + \vartheta^2 = \sqrt{135.246}, \quad \xi^2 - \eta^2 - \zeta^2 + \vartheta^2 = \sqrt{136.245},$$

$$\xi^2 - \eta^2 + \zeta^2 - \vartheta^2 = \sqrt{145.236}, \quad \xi^2 + \eta^2 - \zeta^2 - \vartheta^2 = \sqrt{146.235},$$

$$2(\xi\zeta + \eta\vartheta) = \sqrt{124.356}, \quad 2(\xi\vartheta + \eta\zeta) = \sqrt{125.346}, \quad 2(\xi\eta + \zeta\vartheta) = \sqrt{134.256},$$

$$2(\xi\zeta - \eta\vartheta) = \sqrt{123.456}; \quad 2(\xi\vartheta - \eta\zeta) = \sqrt{126.345}, \quad 2(\xi\eta - \zeta\vartheta) = \sqrt{156.234},$$

when, *e. g.* 135.246 denote $(x_1 - x_3)(x_3 - x_5)(x_5 - x_1)(x_2 - x_4)(x_4 - x_6)(x_6 - x_2)$.

In regard to the isomorphism between the senary group of 360 permutations of the x's and the quaternary group of the linear transformations of the point co-ordinates of the double points (disregarding for the moment the changes of sign which arise from even changes of sign of the x's of the fundamental complexes), Prof. Klein noted in one of his lectures that this might be a holoedric or hemiedric isomorphism, since the change of sign of all the point co-ordinates leaves the line co-ordinates unchanged, so that it might be that to every operation of the senary group two operations of the quaternary group, differing only in sign, correspond. That the isomorphism is hemiedric is evident from the following consideration: For every even permutation of the x's there will be either one or two corresponding linear substitutions of the ξ, η, ζ, ϑ, according as the isomorphism is holoedric or hemiedric. In particular, for the identical substitutions of the x's we shall have either simply the identical substitution of the ξ, η, ζ, ϑ, or this and a second substitution consisting simply in the change of sign of ξ, η, ζ, ϑ. If, therefore, we can simultaneously change the sign of all forms of the Borchardt moduli by a series of permutations of the x's, which result in each x's returning to its original position, the isomorphism will be hemiedric, and conversely, if the latter be the case, such a set of permutations can be found. The quantities x being entirely arbitrary, we select any convenient values of x_2, x_5 and x_6 (no two of which shall coincide) and take x_1 and x_3 and x_4 very near each other. (We may conveniently represent the 6 x's by six points in a plane.) Consider now the quantity: $\sqrt{(135)(246)}$. If x_1 and x_3

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* Borchardt: Abh. d. Berliner Akademie, 1878.

describe continuous paths, so that finally x_1 becomes x_3 and x_3 becomes x_1, this square root will have been multiplied by $\pm i$. A repetition of this operation will give us $-\sqrt{(135)(246)}$. The quantity $\sqrt{(136)(245)}$ will undergo the same transformation. The other two square roots $\sqrt{(145)(236)}$ and $\sqrt{(146)(235)}$ will be unchanged. Again, if we permute x_1 and x_4 in the same way, the latter two square roots will change sign, while the former two remain unchanged. By an identical permutation of the x's we can therefore change the sign of ξ^2, η^2, ζ^2 and ϑ^2 simultaneously. Then ξ, η, ζ and ϑ will each have been multiplied by $\pm i$. A repetition of the operation will change the sign in each case. To every operation of the senary group will correspond the two operations of the quaternary linear group, so that the latter group consists of 32.360 substitutions.

Camille Jordan,* by a process nearly the inverse of the one just given, but which does not employ the line geometry, has exhibited geometrically that it is possible to obtain a group of linear substitutions of four elements which shall be exactly holoedrically isomorphic with the 360 even permutations of six elements. Thus the 16 double points of the Kummer surface satisfy an equation of degree 16, of which the group is determined readily from the fact that the 16 points lie on 16 planes, 6 points in each plane, and 6 planes through each point. This group consists of 16.720 substitutions, and to it corresponds holoedrically a linear group of 16.720 collineations of space. Corresponding functions are then constructed and grouped in sets of 16, each set representing the six double points in a plane, these six sets being permuted among each other. There must be 720 of these permutations, since any interchange of the planes interchanges the double points, and conversely, to every one of these 720 permutations of the six points in a plane will correspond one, and only one, of the 720 collineations. The two groups are therefore referred holoedrically to each other. It follows also that the two alternating sub-groups are holoedrically related. I have introduced this result of Jordan's because it has an immediate bearing on the construction of functions which shall undergo the operations of our linear group. Klein† has shown that if we have two linear groups, one affecting n variables, and containing N substitutions, the other affecting μ variables, and containing $\frac{N}{\nu}$ substitutions, then, if isomorphism exists between these groups, we can construct rational, homogeneous, integral functions of the x's which undergo the substitutions of the y's. As a particular case, the linear substitutions may be merely the permutations of the x's. It appears, therefore, that in the present case we can obtain rational, homogeneous, integral functions of the x's which shall undergo one group of linear substitutions when the x's are permuted. Klein shows in the same article how actually to construct such functions, but I am not aware that this construction has actually been carried through for the present case.

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*Camille Jordan: Traité, etc., p. 313.

†Klein: Ueber die Auflösung gewisser Gleichungen, etc., Bd. XV of the Math. Ann.

The Borchardt moduli are therefore, in two respects, by no means the simplest functions which could be used. They are irrational to a considerable degree, and their group is 32 — 1-edric isomorphic with the alternating group of the x's, and even after the adjunction of the geometry of the Kummer surface we have a meriedric isomorphism. They are, however, well-known functions which have a bearing on various mathematical theories, and they are in a very compact form. Moreover, a set of transformations will readily be found for passing from the formulae which may be obtained from them to any other system of quantities which may afterwards be obtained.

Taking, then, the Borchardt moduli as the fundamental functions, we can immediately obtain a system of linear differential equations with coefficients rational in the coefficients of the given equation and the $\sqrt{\Delta}$, and satisfied by these moduli. These are at once formally given by the following method:

We construct the determinant in which the derivatives are taken with respect to any coefficient of the given equation of the sixth degree:

$$\begin{vmatrix} \omega & \xi & \eta & \zeta & \vartheta \\ \omega' & \xi' & \eta' & \zeta' & \vartheta' \\ \omega'' & \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \omega''' & \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \omega^{\text{IV}} & \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix}.$$

This will evidently be 0 if ω equals ξ, η, ζ or ϑ.

The desired differential equation is got by expanding this and equating to 0:

$$\omega^{\text{IV}} \begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi' & \eta' & \zeta' & \vartheta' \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \end{vmatrix} + \omega''' \begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi' & \eta' & \zeta' & \vartheta' \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix} + \omega'' \begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi' & \eta' & \zeta' & \vartheta' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix}$$

$$+ \omega' \begin{vmatrix} \xi & \eta & \zeta & \vartheta \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix} + \omega \begin{vmatrix} \xi' & \eta' & \zeta' & \vartheta' \\ \xi'' & \eta'' & \zeta'' & \vartheta'' \\ \xi''' & \eta''' & \zeta''' & \vartheta''' \\ \xi^{\text{IV}} & \eta^{\text{IV}} & \zeta^{\text{IV}} & \vartheta^{\text{IV}} \end{vmatrix} = 0.$$

These sub-determinants will evidently only be multiplied by the same constant factor for all when the ξ, η, ζ, ϑ are linearly transformed, *i. e.* when the x's are permuted an even number of times. Their ratios are therefore unchanged by any even permutations of the x's. From this we cannot, however, infer directly that these ratios are rational functions of the coefficients of the proposed equation and of the square root of the discriminant, since we are not certain that the determinant ratios are rational in the x's, while the theory of substitutions deals primarily only with such functions as are rational in the roots. Precisely the same process, however, as that of the theory of substitutions in the limited sense of the word will demonstrate that our determinant ratios are rational in the coefficient and the square root of the discriminant. Thus let $\phi_1\phi_2 \ldots \phi_\lambda$ be the values of the determinant ratios and form

$$\phi_1 + \phi_2 \ldots + \phi_\lambda = A_0,$$
$$\phi_1\sqrt{\Delta} \pm \phi_2\sqrt{\Delta} + \ldots \pm \phi_\lambda\sqrt{\Delta} = A_1, \text{ etc.}$$

From these we can get ϕ_1 as a rational function of the A's and $\sqrt{\Delta}$. From which again it is apparent that $\lambda = 2$. This is a perfectly general proposition. Any two algebraic functions, whether rational in the roots or not, belonging to the same group, are rational functions of each other and of the coefficients of the proposed equation.

To solve the equation of the sixth degree, we must first remove the coefficients of the terms x^5 and x^4, form the corresponding differential equation with respect to each of the four remaining coefficients, and actually calculate the ratios of the determinants which appear as coefficients in these differential equations.

Actually to differentiate ξ, η, ζ, ϑ four times, substitute the results in the 5 determinants, and reduce the result, removing all common factors from numerator and denominator, etc., would be nearly impossible, as one will readily perceive on remembering that any function has 6 first, 21 second, 56 third, and 126 fourth derivatives with respect to six quantities, and that we must first differentiate the ξ, η, ζ, ϑ with respect to the x's and afterwards the x's with respect to the coefficients of the proposed equation. It is therefore necessary to have recourse to infinite series. We must expand the roots in terms of the coefficients, substitute these values in the Borchardt moduli, and obtain them also as infinite series, which are then to be substituted in the determinant ratios, and the latter obtained as rational functions of the coefficients, and of the square root of the discriminant.

Here we meet with a certain technical difficulty. The coefficients of our differential equations are, to be sure, rational functions of the coefficients of the given equation and of the $\sqrt{\Delta}$. But they are not integral functions. From the final infinite series we shall therefore be unable to separate numerator and denominator. There is no method of determining *a priori* the roots and poles of these functions, and any attempt to determine them would necessarily be a tentative process of which the success would be highly improbable. Again, if we attempt to avoid this difficulty by considering each determinant by itself, we are checked in two ways. First, the single determinants may or may not be rational, after the adjunction of the $\sqrt{\Delta}$, according as the constant multiplier is unity or not. As a matter of fact, we shall find later that it is unity in all cases. And further, we have the former difficulty, that the determinants themselves are not integral functions of the coefficients. The former difficulty is easily got over as follows: Every even permutation of the x's multiplies any one of the determinants by some constant factor, which may in particular cases be unity. All permutations of the x's, therefore, give the determinant a set of 32.720 values, all of which are constant multiples of any one of them (equality being regarded as a particular case). All these values of the determinant have, therefore, the same group of substitutions, which will therefore be a self-conjugate sub-group of the alternating group of six elements. But there are only two such sub-groups, namely, the alternating group itself and the group consisting of the identical substitution. The determinant therefore belongs to the entire group of 720 substitutions and is rational, or it belongs to the alternating group and is rational after the adjunction of the square root of the discriminant, or it belongs to the identical substitution and is 32.720-valued. That the last does not occur is evident from the consideration of the group itself, one of the substitutions of which interchanges ξ and ζ, and a second η and ϑ, the product of the two, therefore, leaving each determinant unchanged. The determinants are, therefore, of the form $\frac{S_1+S_2\sqrt{\Delta}}{S_3+S_4\sqrt{\Delta}}$, where S_1, S_2, S_3, S_4 denote integral rational functions of the coefficients of the proposed equation. To determine more completely the nature of the determinant, we must observe how it behaves when the x's undergo odd permutations. On the one hand, these will all convert $\frac{S_1+S_2\sqrt{\Delta}}{S_3+S_4\sqrt{\Delta}}$ into $\frac{S_1-S_2\sqrt{\Delta}}{S_3-S_4\sqrt{\Delta}}$. On the other hand, if we interchange, say, x_1 and x_2, ξ^2 and η^2 are unchanged, while ζ^2 and ϑ^2 are each multiplied by -1. The 10 conditions between the

$\xi, \zeta, \eta, \vartheta$ are satisfied if we put for $\xi, \eta, \zeta, \vartheta$, $\xi, \eta, i\zeta, i\vartheta$. The interchange of x_1 and x_2, therefore, changes the sign of each determinant. That these two results may agree, we must have $S_1S_3 + S_z S_y \Delta = 0$. From this either $\frac{S_1+S_2\sqrt{\Delta}}{S_3+S_4\sqrt{\Delta}} = \frac{S_2\sqrt{\Delta}}{S_3}$, or $S_1 = S_4 = 0$, or $S_2 = S_3 = 0$. In any case we see that our determinants are of the form $\frac{S}{S'\sqrt{\Delta}}$, where the S's are rational integral functions of the coefficients.

The second difficulty would be more formidable if the nature of the Borchardt moduli themselves were not in our favor. For it will appear that for these moduli the function S' is a power of the discriminant, so that the denominator is fully determined from the start, except that the index of the power is unknown. This simplification will also occur, as we shall see, in the case of any of the systems of moduli proposed by Klein, it being a sufficient condition for rational moduli that they shall be integral. That it is the case with the Borchardt moduli appears in actually differentiating. Suppose, for instance, we consider the differential equation with respect to the coefficient of x. For the first derivative of ξ we have

$$\frac{d\xi}{df} = \frac{d\xi}{dx_1}\frac{dx_1}{df} + \frac{d\xi}{dx_2}\frac{dx_2}{df} + \frac{d\xi}{dx_3}\frac{dx_3}{df} + \frac{d\xi}{dx_4}\frac{dx_4}{df} + \frac{d\xi}{dx_5}\frac{dx_5}{df} + \frac{d\xi}{dx_6}\frac{dx_6}{df},$$

and if our equation be

$$fx = x^6 + dx^3 + ex^2 + fx + g = 0, \quad \frac{dx_i}{df} = -\frac{x_i}{f'x_i} = -\frac{x_i}{\pi(x'_j - x_i)},$$

where j takes every value from 1 to 6 except i. Again, each of the derivatives of ξ with respect to the x_i will evidently contain in the denominator only powers of ξ and products of differences of roots. The denominator of $\frac{d\xi}{df}$ will, therefore, be composed entirely of products of differences of roots and powers of ξ, and it is evident that the same will be true for all the derivatives of any order of $\xi, \eta, \zeta, \vartheta$ taken with respect to any coefficient. For future reference I give the six first derivatives of ξ with respect to the x_i:

$$\begin{aligned}\frac{d\xi}{dx_1} = \frac{1}{2\xi}\Bigg\{ & \frac{(x_3 - x_5)(x_3 - 2x_1 + x_5)(x_2 - x_4)(x_4 - x_6)(x_6 - x_2)}{2\sqrt{(x_1 - x_3)(x_3 - x_5)(x_5 - x_1)(x_2 - x_4)(x_4 - x_6)(x_6 - x_2)}} \\ & + \frac{(x_4 - x_5)(x_4 - 2x_1 + x_5)(x_2 - x_3)(x_3 - x_6)(x_6 - x_2)}{2\sqrt{(x_1 - x_4)(x_4 - x_5)(x_5 - x_1)(x_2 - x_3)(x_3 - x_6)(x_6 - x_2)}} \\ & - \frac{(x_3 - x_6)(x_3 - 2x_1 + x_6)(x_2 - x_4)(x_4 - x_5)(x_5 - x_2)}{2\sqrt{(x_1 - x_3)(x_3 - x_6)(x_6 - x_1)(x_2 - x_4)(x_4 - x_5)(x_5 - x_2)}} \\ & + \frac{(x_4 - x_6)(x_4 - 2x_1 + x_6)(x_2 - x_3)(x_3 - x_5)(x_5 - x_2)}{2\sqrt{(x_1 - x_4)(x_4 - x_6)(x_6 - x_1)(x_2 - x_3)(x_3 - x_5)(x_5 - x_2)}}\Bigg\},\end{aligned}$$

$$\frac{d\xi}{dx_3} = \frac{1}{2\xi}\left\{\frac{(x_5-x_1)(x_5-2x_3+x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)}{2\sqrt{(x_1-x_3)(x_3-x_5)(x_5-x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)}}\right.$$
$$+\frac{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_6-x_2)(x_6-2x_3+x_2)}{2\sqrt{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_2-x_3)(x_3-x_6)(x_6-x_2)}}$$
$$+\frac{(x_6-x_1)(x_6-2x_3+x_1)(x_2-x_4)(x_4-x_5)(x_5-x_1)}{2\sqrt{(x_1-x_3)(x_3-x_6)(x_6-x_1)(x_2-x_4)(x_4-x_5)(x_5-x_2)}}$$
$$\left.+\frac{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_5-x_2)(x_5-2x_3+x_2)}{2\sqrt{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_2-x_3)(x_3-x_5)(x_5-x_2)}}\right\},$$

$$\frac{d\xi}{dx_5} = \frac{1}{2\xi}\left\{\frac{(x_1-x_3)(x_1-2x_5+x_3)(x_2-x_4)(x_4-x_6)(x_6-x_2)}{2\sqrt{(x_1-x_3)(x_3-x_5)(x_5-x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)}}\right.$$
$$+\frac{(x_1-x_4)(x_1-2x_5+x_4)(x_2-x_3)(x_3-x_6)(x_6-x_2)}{2\sqrt{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_2-x_3)(x_3-x_6)(x_6-x_2)}}$$
$$+\frac{(x_2-x_4)(x_2-2x_5+x_4)(x_1-x_3)(x_3-x_6)(x_6-x_1)}{2\sqrt{(x_1-x_3)(x_3-x_6)(x_6-x_1)(x_2-x_4)(x_4-x_5)(x_5-x_2)}}$$
$$\left.+\frac{(x_2-x_3)(x_2-2x_5+x_3)(x_1-x_4)(x_4-x_6)(x_6-x_1)}{2\sqrt{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_2-x_3)(x_3-x_5)(x_5-x_2)}}\right\}.$$

The other three are obtained from these by permuting x_1 and x_2, x_3 and x_4, x_5 and x_6 respectively.

If, now, we expand our determinants, we shall get a fractional function of which the denominator will be composed of products of differences of roots and some power of the product ξ, η, ζ, ϑ. Since $2(\xi\zeta+\eta\vartheta)=\sqrt{(124)(356)}$ and $2(\xi\zeta-\eta\vartheta)=\sqrt{(123)(456)}$, we have

$$\begin{aligned}8\xi\zeta\eta\vartheta &= 124.356-123.456 = 12.56.(24.41.35.63-23.31.45.64)\\ &= 12.34.56\big((x_3x_4-x_1x_2)(x_5+x_6)+(x_5x_6-x_3x_4)(x_1+x_2)+(x_1x_2-x_5x_6)(x_3+x_4)\big).\end{aligned}$$

Since, now, the denominator is either symmetrical or alternating, it is evident that every difference of a pair of roots will occur as often as any other difference. These differences may therefore be grouped together and give some power of the $\sqrt{\Delta}$. There remains over a power of the parenthesis above. This is neither symmetrical nor alternating, but is 30-valued. *The numerator, then, must be of such a nature that this* 30-*valued function cancels out*, and this shows at once that the denominator is simply some power of the square root of the discriminant. That this is an odd power is evident from the above.

For the rational functions proposed by Klein, nothing but differences of the roots would occur in the denominator from the start, and for them the denominator would also be merely a power of the square root of the discriminant.

The knowledge of the nature of our determinants obtained by this process of reasoning suffices to remove all theoretical difficulties in their computation, which is now perfectly straightforward, but extremely long and tedious. It is necessary to determine to what power the square root of the discriminant enters in the denominator. From this the degree of the numerator is easily ascertained, and the numerator itself is got at once by multiplying our final infinite series by the denominator.

In the following computations, I have aimed at complete accuracy above everything else. Every step has been thoroughly tested by parallel methods, so that there is scarcely any possibility of error in the determination of the infinite series for ξ, η, ζ, ϑ. Errors may have crept in in the final determination of the coefficients in their rational form. These will readily be detected when the coefficients have been finally determined in full. Although the value which these determinations may have depends entirely on their accuracy, I have not thought it best to give them in full, on account of their great length. I have deposited a copy of them in the library of Harvard University. They may be divided into four parts: 1st. The calculation of the roots in terms of the coefficients. This I have carried out, for the case where the coefficients x^5, x^4, x^3 and x^2 are 0, to eleven terms, as follows:

$$x = \varepsilon - \frac{\varepsilon^2}{6}\alpha - \frac{\varepsilon^3}{24}\alpha^2 - \frac{\varepsilon^4}{81}\alpha^3 - \frac{91\varepsilon^5}{3^5.2^7}\alpha^4 + \frac{4301\varepsilon}{3^8.2^{10}}\alpha^6 + \frac{11\varepsilon^2}{3^9}\alpha^7 + \frac{11\varepsilon^3}{2^{15}}\alpha^8$$
$$+ \frac{247\varepsilon^4}{3^{13}}\alpha^9 + \frac{60911435\varepsilon^5}{3^{14}.2^{18}}\alpha^{10} + 0\alpha^{11}.$$

In the accompanying table I have given only the test of the accuracy of the determination, by actually substituting the value of x in the equation, but I obtained the coefficient in another way. It will be noted that every sixth coefficient, beginning with the fifth, vanishes, which fact is of considerable assistance in the computation, and is a good test of correctness. In getting the expansion, I have supposed the equation to be divided through by $-\delta$, and have put for x, $\frac{x}{(-\delta)^{\frac{1}{6}}}$, and for α $\frac{\alpha}{(-\delta)^{\frac{5}{6}}}$, so that the Borchardt moduli must be multiplied finally by $\sqrt[4]{-\delta}$ and for α we must put $\frac{\alpha}{(-\delta)^{\frac{5}{6}}}$.

2d. The determination of the products (135)(246), etc. This is much facilitated by the following considerations: The first terms of the six series for the roots are 1, ε, ε^2, -1, $-\varepsilon$, $-\varepsilon^2$, in order. Consider now the equation of which the roots are ε^3 times the present ones. This equation will be of the form $x^6 - \varepsilon a x - 1 = 0$. The roots of this equation will be got by putting $-\varepsilon\alpha$ for α in the six series above. If now we calculate for this new system of roots the products (135)(246), etc., these will evidently be equal to the former products, but in a different order; for each root has been moved forward two places in the order 123456, the first being multiplied by ε^2 and becoming the third so far as the first term is concerned. Having obtained (145)(236), we get therefore (146)(235) and (136)(245) by putting for x, $-\varepsilon\alpha$ and $\varepsilon^2\alpha$ respectively. It is therefore only necessary to calculate (135)(246) and (145)(236), and in case of the former, (246) is at once obtainable from (123) by putting $-\varepsilon^2\alpha$ for α. We have

$$(135) = -3(\varepsilon+\varepsilon^2)\left(1 - \frac{7}{216}\alpha^3 + \frac{1957}{3^6.\,2^{10}}\alpha^6 + \frac{240691}{3^9.\,2^{15}}\alpha^9 \ldots.\right),$$

$$(246) = -3(\varepsilon+\varepsilon^2)\left(-1 - \frac{7}{216}\alpha^3 - \frac{1957}{3^6.\,2^{10}}\alpha^6 + \frac{240691}{3^9.\,2^{15}}\alpha^9 \ldots.\right),$$

$$\begin{aligned}(145) = 2 - 2\varepsilon^2 &+ \frac{2(1+\varepsilon)}{3}\alpha + \frac{\varepsilon-6}{12}\alpha^2 - \frac{7-14\varepsilon}{2^2.\,3^4}\alpha^3 - \frac{350-91\varepsilon}{3^5.\,2^6}\alpha^4 \\ &- \frac{196-98\varepsilon}{3^6.\,2^7}\alpha^5 + \frac{35814-17613\varepsilon}{3^8.\,2^9}\alpha^6 - \frac{34130+34130\varepsilon}{3^9.\,2^{10}}\alpha^7 \\ &+ \frac{3210518-358723\varepsilon}{3^{10}.\,2^{14}}\alpha^8 + \frac{14055494-28110988\varepsilon}{3^{13}.\,2^{15}}\alpha^9 \\ &+ \frac{214802770-57182027\varepsilon}{3^{14}.\,2^{17}}\alpha^{10} + \frac{73539718-36769859\varepsilon}{3^{15}.\,2^{17}}\alpha^{11},\end{aligned}$$

$$\begin{aligned}(236) = 2(\varepsilon+1) &+ \frac{2-4\varepsilon}{3}\alpha + \frac{6\varepsilon-1}{12}\alpha^2 - \frac{7-14\varepsilon}{9.36}\alpha^3 + \frac{350-259\varepsilon}{3^5.2^6}\alpha^4 \\ &+ \frac{98-196\varepsilon}{3^6.\,2^7}\alpha^5 + \frac{18201+17613\varepsilon}{3^8.\,2^9}\alpha^6 - \frac{34130-68260\varepsilon}{3^9.\,2^{10}}\alpha^7 \\ &+ \frac{358723-3210518\varepsilon}{3^{10}.\,2^{14}}\alpha^8 + \frac{14055494-28110988\varepsilon}{3^{13}.\,2^{15}}\alpha^9 \\ &- \frac{214802770-157620743\varepsilon}{3^{14}.\,2^{17}}\alpha^{10} - \frac{36769859-73539718\varepsilon}{3^{15}.\,2^{17}}\alpha^{11},\end{aligned}$$

$$135.246 = -27\left(-1 - \frac{1565}{3^6.\,2^9}\alpha^6 \ldots.\right) = 27 + \frac{1565}{3^3.\,2^9}\alpha^6 + \ldots.,$$

$$\begin{aligned}145.236 = 12 &+ \frac{5(1-\varepsilon)}{6}\alpha^2 + \frac{20(\varepsilon+\varepsilon^2)}{27}\alpha^3 - \frac{2352}{3^3.\,2^5}\alpha^4 + \frac{30095}{3^7.\,2^8}\alpha^6 - \frac{112915(1-\varepsilon)}{3^7.\,2^{13}}\alpha^8 \\ &- \frac{118285(\varepsilon^2+\varepsilon)}{3^{12}.\,2^5}\alpha^9 + \frac{1512905\varepsilon}{3^8.\,2^{16}}\alpha^{10}.\end{aligned}$$

The disappearance of terms in regular order again furnishes a valuable control of the accuracy up to this point.

3d. The extraction of the square roots of the products above:

$$\sqrt{(x_1-x_4)(x_4-x_5)(x_5-x_1)(x_2-x_3)(x_3-x_6)(x_6-x_2)}$$

$$= 2\sqrt{3} + \frac{5(1-\varepsilon)}{24\sqrt{3}}\alpha^2 + \frac{20(\varepsilon+\varepsilon^2)}{108\sqrt{3}}\alpha^3 - \frac{445\varepsilon}{3^2.\,2^8\sqrt{3}}\alpha^4 - \frac{25(\varepsilon+1)}{3^5.\,2^4\sqrt{3}}\alpha^5 + \frac{217205}{3^7.\,2^{12}\sqrt{3}}\alpha^6$$

$$+ \frac{1975(\varepsilon^2-1)}{3^7.\,2^9\sqrt{3}}\alpha^7 - \frac{13763755(1-\varepsilon)}{3^9.\,2^{18}\sqrt{3}}\alpha^8 - \frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.\,2^{13}\sqrt{3}}\alpha^9$$

$$+ \frac{891732635\varepsilon}{3^{11}.\,2^{22}\sqrt{3}}\alpha^{10} + \frac{452064875(1+\varepsilon)}{3^{14}.\,2^{19}\sqrt{3}}\alpha^{11},$$

$$\sqrt{(x_1-x_3)(x_3-x_6)(x_6-x_1)(x_2-x_4)(x_4-x_5)(x_5-x_2)}$$

$$= 2\sqrt{3} - \frac{5}{24\sqrt{3}}\alpha^2 + \frac{20(\varepsilon+\varepsilon^2)}{108\sqrt{3}}\alpha^3 + \frac{445}{3^3.\,2^8\sqrt{3}}\alpha^4 + \frac{25(\varepsilon+\varepsilon^2)}{3^5.\,2^4\sqrt{3}}\alpha^5 + \frac{217205}{3^7.\,2^{12}\sqrt{3}}\alpha^6$$

$$- \frac{1975(\varepsilon+\varepsilon^2)}{3^7.\,2^9\sqrt{3}}\alpha^7 + \frac{13763755}{3^9.\,2^{18}\sqrt{3}}\alpha^8 - \frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.\,2^{13}\sqrt{3}}\alpha^9 - \frac{891732635}{3^{11}.\,2^{22}\sqrt{3}}\alpha^{10}$$

$$- \frac{452064875(\varepsilon+\varepsilon^2)}{3^{14}.\,2^{19}\sqrt{3}}\alpha^{11},$$

$$\sqrt{(x_1-x_4)(x_4-x_6)(x_6-x_1)(x_2-x_3)(x_3-x_5)(x_5-x_2)}$$

$$= 2\sqrt{3} + \frac{5(1+\varepsilon^2)}{24\sqrt{3}}\alpha^2 + \frac{20(\varepsilon+\varepsilon^2)}{108\sqrt{3}}\alpha^3 + \frac{445\varepsilon^2}{3^3.\,2^8\sqrt{3}}\alpha^4 + \frac{25(1+\varepsilon^2)}{3^5.\,2^4\sqrt{3}}\alpha^5 + \frac{217205}{3^7.\,2^{12}\sqrt{3}}\alpha^6$$

$$+ \frac{1975(1+\varepsilon)}{3^7.\,2^9\sqrt{3}}\alpha^7 - \frac{13763755\varepsilon}{3^9.\,2^{18}\sqrt{3}}\alpha^8 - \frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.\,2^{13}\sqrt{3}}\alpha^9 - \frac{891732635\varepsilon^2}{3^{11}.\,2^{22}\sqrt{3}}\alpha^{10}$$

$$+ \frac{452064875(\varepsilon^2-1)}{3^{14}.\,2^{19}\sqrt{3}}\alpha^{11},$$

$$\sqrt{(x_1-x_3)(x_3-x_5)(x_5-x_1)(x_2-x_4)(x_4-x_6)(x_6-x_2)} = 3\sqrt{3}\ldots + \frac{1565}{3^4.\,2^{10}\sqrt{3}}\alpha^6,$$

the combining of the results so as to give ξ^2, η^2, ζ^2, ϑ^2:

$$4\xi^2 = 9\sqrt{3} + \frac{5(\varepsilon+\varepsilon^2)}{9\sqrt{3}}\alpha^3 + \frac{273545}{3^6.\,2^{12}\sqrt{3}}\alpha^6 - \frac{11395315(\varepsilon+\varepsilon^2)}{3^{11}.\,2^{13}\sqrt{3}}\alpha^9,$$

$$4\zeta^2 = \sqrt{3} - \frac{5\varepsilon^2}{12\sqrt{3}}\alpha^2 - \frac{5(\varepsilon+\varepsilon^2)}{27\sqrt{3}}\alpha^3 - \frac{445\varepsilon}{3^3.\,2^7\sqrt{3}}\alpha^4 - \frac{25(\varepsilon+1)}{3^5.\,2^3\sqrt{3}}\alpha^5 - \frac{48185}{3^7.\,2^{12}\sqrt{3}}\alpha^6$$

$$+ \frac{1975(\varepsilon^2-1)}{3^7.\,2^8\sqrt{3}}\alpha^7 + \frac{13763755(\varepsilon-1)}{3^9.\,2^{17}\sqrt{3}}\alpha^8 + \frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}2^{13}\sqrt{3}}\alpha^9$$

$$+ \frac{891732635\varepsilon}{3^{11}.\,2^{21}\sqrt{3}}\alpha^{10} + \frac{452064875(1+\varepsilon)}{3^{14}.\,2^{18}\sqrt{3}}\alpha^{11}.,$$

$$4\vartheta^2 = \sqrt{3} - \frac{5}{12\sqrt{3}}\alpha^2 - \frac{5(\varepsilon+\varepsilon^2)}{27\sqrt{3}}\alpha^3 + \frac{445}{3^3.\,2^7\sqrt{3}}\alpha^4 + \frac{25(\varepsilon+\varepsilon^2)}{3^5.\,2^3\sqrt{3}}\alpha^5 - \frac{48185}{3^7.\,2^{12}\sqrt{3}}\alpha^6$$

$$- \frac{1975(\varepsilon^2+\varepsilon)}{3^7.\,2^8\sqrt{3}}\alpha^7 + \frac{13763755}{3^9.\,2^{17}\sqrt{3}}\alpha^8 + \frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.\,2^{13}\sqrt{3}}\alpha^9$$

$$- \frac{891732635}{3^{11}.\,2^{21}\sqrt{3}}\alpha^{10} - \frac{452064875(\varepsilon+\varepsilon^2)}{3^{14}.\,2^{18}\sqrt{3}}\alpha^{11},$$

$$4\eta^2 = \sqrt{3} + \frac{5\varepsilon}{12\sqrt{3}}\alpha^2 - \frac{5(\varepsilon+\varepsilon^2)}{27\sqrt{3}}\alpha^3 + \frac{445\varepsilon^2}{3^3.2^7\sqrt{3}}\alpha^4 + \frac{25(1-\varepsilon^2)}{3^5.2^3\sqrt{3}}\alpha^5 - \frac{48185}{3^7.2^{12}\sqrt{3}}\alpha^6$$
$$+ \frac{1975(\varepsilon+1)}{3^7.2^8\sqrt{3}}\alpha^7 - \frac{13763755\varepsilon}{3^9.2^{17}}\alpha^8 + \frac{11395315(\varepsilon+\varepsilon^2)}{3^{12}.2^{13}\sqrt{3}}\alpha^9$$
$$- \frac{891732635\varepsilon^2}{3^{11}.2^{21}\sqrt{3}}\alpha^{10} + \frac{452064875(\varepsilon^2-1)}{3^{14}2^{18}\sqrt{3}}\alpha^{11},$$

and the extraction of the square roots in each case:

$$\xi = \frac{3}{2}\sqrt[4]{3} + \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^3 + \frac{99715}{3^6.2^{14}\sqrt[4]{27}}\alpha^6 - \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9,$$

$$\zeta = \frac{1}{2}\sqrt[4]{3} - \frac{5\varepsilon^2}{48\sqrt[4]{27}}\alpha^2 - \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^3 - \frac{395\varepsilon}{3^3.2^9\sqrt[4]{27}}\alpha^4 + \frac{99715}{3^7.2^{14}\sqrt[4]{27}}\alpha^6 + \frac{479615\varepsilon^2}{3^6.2^{19}\sqrt[4]{27}}\alpha^8$$
$$+ \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9 + \frac{93445715\varepsilon}{3^9.2^{23}\sqrt[4]{27}}\alpha^{10},$$

$$\vartheta = \frac{1}{2}\sqrt[4]{3} - \frac{5\alpha^2}{48\sqrt[4]{27}} - \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^3 + \frac{395}{3^8.2^9\sqrt[4]{27}}\alpha^4 + \frac{99715}{3^7.2^{14}\sqrt[4]{27}}\alpha^6 + \frac{479615}{3^6.2^{19}\sqrt[4]{27}}\alpha^8$$
$$+ \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9 - \frac{93445715}{3^9.2^{23}\sqrt[4]{27}}\alpha^{10},$$

$$\eta = \frac{1}{2}\sqrt[4]{3} + \frac{5\varepsilon\alpha^2}{48\sqrt[4]{27}} - \frac{5(\varepsilon+\varepsilon^2)}{108\sqrt[4]{27}}\alpha^8 + \frac{395\varepsilon^2}{3^3.2^9\sqrt[4]{27}}\alpha^4 + \frac{99715}{3^7.2^{14}\sqrt[4]{27}}\alpha^6 - \frac{479615\varepsilon}{3^6.2^{19}\sqrt[4]{27}}\alpha^8$$
$$+ \frac{402845(\varepsilon+\varepsilon^2)}{3^{12}.2^{10}\sqrt[4]{27}}\alpha^9 - \frac{93445715\varepsilon^2}{3^9.2^{23}\sqrt[4]{27}}\alpha^{10}.$$

Here again terms vanish in regular order; namely, for ξ, all terms except those whose order is divisible by 3, and for ζ, η, ϑ, the 5th, 7th, 11th, 13th, etc. Further, the relation between the coefficients of ξ and those of the corresponding terms of ζ, η, ϑ, is a valuable check. These I have tested by means of $2(\xi\zeta - \eta\vartheta) = \sqrt{123.456}$, which ensures the correctness of the values of the $\xi\eta\zeta\vartheta$.

4th. The determination of the coefficients of the differential equation. First of all, the degree of the denominator must be obtained. The values of α for which two roots are equal are got by equating the discriminant to $0.6^6.\alpha^6 - 5^5 = 0$.* This gives 6 finite values of α, each of which is, in our case, a point of the first order; *i. e.* only two roots are equal for any one of these values of α. We may therefore expand the roots in ascending powers of $\alpha - \alpha_0$, if α_0 be one of these values. For those roots which do not coincide we shall have only integral powers, while for the two coincident roots the $\frac{1}{2}$ powers of $\alpha - \alpha_0$ will occur. If these series be introduced into the Borchardt moduli, as may legitimately be done, we shall have the latter expanded in a system of convergent series, in which the $\frac{1}{4}$ and $\frac{1}{8}$ powers may occur, *but no negative powers.* If there be written $a_0 + a_1(\alpha - \alpha_0)^{\frac{1}{8}} + \ldots . \; b_0 + b_1(\alpha - \alpha_0)^{\frac{1}{8}} + \ldots$, etc., and if these series be differentiated and substituted in the determinants, we may determinate the highest power to which $\alpha - \alpha_0$ appears in the denominator, and this will be the upper limit of the power of any factor $x_1 - x_2$ in the denominator, and therefore of the square root of the discriminant to be determined. This matter is considerably simplified by the following considerations: If two roots are equal, say x_1 and x_2, since $x_1 - x_2$ is not a factor of any of the square roots $\sqrt{(135)(246)}$, etc., occurring in the Borchardt moduli, these square roots will involve only $\frac{1}{2}$ powers and integral powers of $(\alpha - \alpha_0)$. Again, if two roots are equal, two of the moduli vanish, while two do not. The two which do not vanish will contain only $\frac{1}{2}$ powers and integral powers of $\alpha - \alpha_0$, while the other two may also involve $\frac{1}{4}$ powers. Consider now the first coefficient in the differential equation for α. The numbers in the determinant denote the highest negative power of $\alpha - \alpha_0$ which occurs in the corresponding infinite series:

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* See Salmon's Higher Algebra, p. 263.

$$\begin{vmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{2} & -\frac{3}{2} & -\frac{7}{4} & -\frac{7}{4} \\ -\frac{5}{2} & -\frac{5}{2} & -\frac{11}{4} & -\frac{11}{4} \end{vmatrix}.$$

The highest negative power of $\alpha - \alpha_0$ which will occur in the development of this determinant will be the power whose index is $\frac{11}{4} + \frac{7}{4} + \frac{1}{2} = 5$. But since the square root of the discriminant must occur to an odd power, we see that $\Delta^{\frac{9}{2}}$ will contain the denominator as a factor.

But ξ, η, ζ, ϑ are homogeneous functions of the x's of degree $\frac{3}{2}$, and α is homogeneous of degree 5. Hence the determinant just considered will be of degree $4 . \frac{3}{2} - 5 - 10 - 15 = -24$. Since the square root of the discriminant is of degree 15 in the roots, it must enter at least to the third power in the denominator, which is therefore either $\Delta^{\frac{3}{2}}$, $\Delta^{\frac{5}{2}}$, $\Delta^{\frac{7}{2}}$ or $\Delta^{\frac{9}{2}}$. Having now expanded the determinant in an infinite series, we may try these denominators successively. First, if the denominator is $\Delta^{\frac{3}{2}}$, the numerator will be of degree $3 . 15 - 24 = 21$.

The only combination of α and δ which is of this degree is $\alpha^3\delta$; the numerator could therefore contain only this one term, and all following terms must cancel. This is not the case. As I have obtained only two terms of the infinite series in $\frac{\alpha}{\delta}$, it is impossible to proceed further in this way. I have therefore begun at the other end of the coefficient, and have obtained a development of the moduli in terms of $\frac{\delta}{\alpha}$ instead of $\frac{\alpha}{\delta}$. This will serve as a very perfect test of the work ultimately, and the consideration of the series in this form is instructive as showing that the term which would contain the lowest power of δ must vanish in every coefficient of the differential equation with respect to α. For consider the four series in $\frac{\delta}{\alpha}$, $\frac{a_0}{\alpha^\mu} + a_1, \frac{\delta}{\alpha^\nu} +$ etc., $\frac{b_0}{\alpha^\mu} + b_1, \frac{\delta}{\alpha^\nu}$, etc. Differentiate these, and form the determinant and expand in powers of $\frac{\delta}{\alpha}$. Evidently the constant term will be

$$a_0b_0c_0d_0 \left| \begin{array}{ccc} 1 & 1 & \\ \mu - 1 & \mu - 1 & \\ (\mu - 2)(\mu - 1) & (\mu - 2)(\mu - 1), & \text{etc.,} \\ (\mu - 3)(\mu - 2)(\mu - 1) & (\mu - 3)(\mu - 2)(\mu - 1) & \end{array} \right| ,$$

and is therefore 0. The same will be true for every coefficient which is not obtained from four sets of coefficients of different powers of δ. Evidently the first power of δ for which this is possible is the sixth power, which can be obtained by taking the constant term from one set, the term of 1st degree from another, and the terms of 2d and 3d degrees from the other two. *The lowest power of δ which appears in any coefficient of the differential equation for α will be the sixth power.*

We can now test the hypothesis $\Delta^{\frac{5}{2}}$, since, in this case, the numerator contains only the term $\alpha^3\delta^6$, and this also fails. Finally, I have actually calculated the coefficient of δ^6 in this coefficient by the process indicated above, and which evidently requires only three terms of the series in $\frac{\delta}{\alpha}$. This coefficient having been determined, $\Delta^{\frac{7}{2}}$ can be tested, and it too fails. It follows, therefore, that the denominator of the first coefficient is $\Delta^{\frac{9}{2}}$. This being known, it appears at once that the following denominators are $\Delta^{\frac{11}{2}}$, $\Delta^{\frac{13}{2}}$, $\Delta^{\frac{15}{2}}$, $\Delta^{\frac{17}{2}}$. The numerators can then be written at once with arbitrary coefficients. Thus that of the first coefficient is $p_1\alpha^3\delta^{16} + p_2\alpha^9\delta^{11} + p_3\alpha^{15}\delta^6$. The last coefficient is determined in each case from the $\frac{\delta}{\alpha}$ series, the first two from the $\frac{\alpha}{\delta}$ series.

(In determining the coefficient I have used both the method of undetermined coefficients and that of the expansion of the determinant.)

This gives us at once all the terms of the first coefficient. We notice now that the second coefficient is the derivative of the first, so that all its terms are known. In the other coefficients, two, three and four terms respectively are still wanting. To obtain them both, the series must be extended further. These further extensions will also check each other as they meet, ensuring absolute accuracy in the result. The differential equation for α is, so far as I have determined it,

$$A\frac{d^4\xi}{d\alpha^4} - B\frac{d^3\xi}{d\alpha^3} + C\frac{d^2\xi}{d\alpha^2} - D\frac{d\xi}{d\alpha} + E = 0, \text{ where}$$

$$A = \frac{-3^{21}.2^{17}.5^3.79a^3\delta^{16} + 3^{15}.2^8.5^8.4649a^9\delta^{11} \pm 19.3^{11}.5^{13}.2^2.a^{15}\delta^6}{\Delta^{\frac{9}{2}}}\,i,$$

$$B = \frac{dA}{da} = \frac{-3^{28}.2^{23}.5^3.79a^2\delta^{21} + 3^{22}.2^{14}.5^8.8891a^8\delta^{16} + 3^{17}.2^8.5^{13}(9298 \pm 285)\,a^{14}\delta^{11} \pm 19.5^{18}.2^4.3^{12}a^{20}\delta^6}{\Delta^{\frac{11}{2}}}\,i,$$

$$C = \frac{-5^3.79.3^{34}.2^{30}.a\delta^{26} - 5^7.3^{27}.2^{21}.6623a^7\delta^{21} + \ldots. \pm 19.3^{11}.2^2.5^{23}.a^{25}.\delta^6}{\Delta^{\frac{13}{2}}}\,i,$$

$$D = \frac{125.79.3^{40}.2^{36}a\delta^{31} + 5^7.3^{33}.2^{26}.393413a^7\delta^{26} + \ldots. \pm 19.3^{11}.2^4.5^{27}\ a^{35}.\delta^6}{\Delta^{\frac{15}{2}}},$$

$$E = \frac{3^{39}.2^{32}.125.79.99715a^5\delta^{36} + \ldots. \pm 19.3^{16}.2^8.5^{29}a^{41}\delta^6}{\Delta^{\frac{17}{2}}}.$$

In the final results, the coefficients are by no means so irregular as at first appears. The powers of 2 and 3 may readily be combined with the variable, since their variation is so regular. This I have not cared to do until I could obtain the complete coefficients.

The further theory of these equations, which I intend to treat in future papers, will consist in the complete determination of all the coefficients, and the transformation into similar equations, referred to the four absolute invariants of the binary sextic as independent variables. This transformation can probably be effected by means of the "Typische Darstellung" of Clebsch.* The relation of the equations to the theory of the bisection of the hyperelliptic integrals of deficiency 2 constitutes a theory by itself. Finally, the theory of these equations should be studied in the manner which Schwarz has made so well known in the case of the linear differential equations of the second order with rational coefficients, and ultimately I hope to study them in connection with that latest of Sylvester's great methods—The Theory of Reciprocants.

CAMBRIDGE, June, 1886.

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* Clebsch: Binöm. Formen, p. 455.